NORTH SYDNEY GIRLS HIGH SCHOOL

HSC Mathematics Extension 1
Assessment Task 2
Term 1, 2016

Name: ______________________________ **Mathematics Class:**__________

Student Number: ________________

Time Allowed: **55 mins + 2 mins reading time**
Total Marks: 36
Section I - Pages 2 - 4
5 Marks
Attempt Questions 1- 5
Answer on Multiple Choice Answer Sheet

Section II - Pages 5 - 8
31 Marks
Attempt Questions 6 - 8
Show all necessary working

Question	Q1	Q2-3	Q4-5	Q6 a) b) d)	Q6 c)	Q7 a(i) a(iii) b	Q7 a(ii)	Q8 a)	Q8 b) c)	Total
HE2					/3		/3			/6
HE6	/1		/2	/8				/3		/14
HE7		/2				/7			/7	/16
										36

Section I

5 Marks
Attempt Questions 1 – 5
Allow about 7 minutes.
Use the multiple choice answer sheet for questions 1 - 5

1. Using $u = x^3 + 5,$ which of the following is equivalent to $\int_{-1}^{1} \frac{x^2 dx}{\sqrt{x^3+5}}$?

(A) $3\int_{-1}^{1} u^{-\frac{1}{2}} du$

(B) $\frac{1}{3}\int_{-1}^{1} u^{-\frac{1}{2}} du$

(C) $3\int_{4}^{6} u^{-\frac{1}{2}} du$

(D) $\frac{1}{3}\int_{4}^{6} u^{-\frac{1}{2}} du$

2. Consider the series:

$$\frac{5}{2x} + \frac{5}{2x^2} + \frac{5}{2x^3} + \ldots$$

What are the possible values for *x* so that the series will have a limiting sum?

(A) $-1 < x < 1 \quad x \neq 0$

(B) $-\frac{1}{2} < x < \frac{1}{2} \quad x \neq 0$

(C) $x < -1, \quad x > 1$

(D) $x < -\frac{1}{2}, \quad x > \frac{1}{2}$

3. What is the value of $\sum_{n=2}^{10}(2-5n)$?

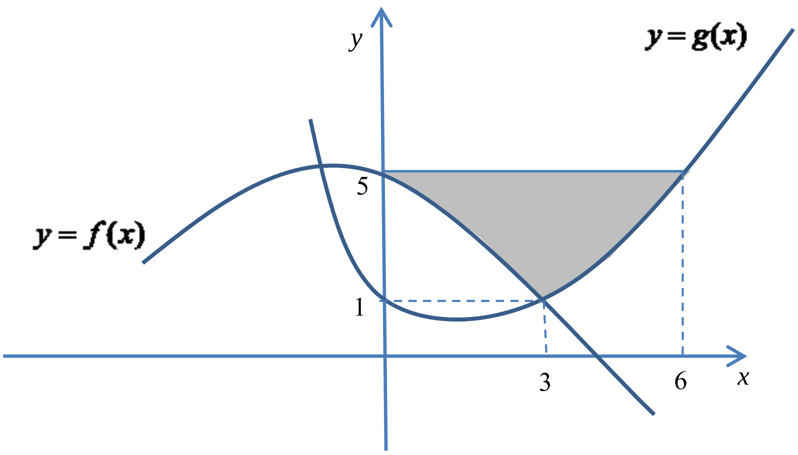
(A) -56

(B) -224

(C) -252

(D) -504

4. Consider the graphs of the two functions $y = f(x)$ and $y = g(x)$ below.



Which of the following definite integrals is equal to the area of the shaded region?

(A) $\int_1^5 \left(g(x) - f(x)\right) dx$

(B) $\int_1^5 \left(f(x) - g(x)\right) dx$

(C) $\int_0^3 \left(5 - g(x)\right) dx + \int_3^6 \left(5 - f(x)\right) dx$

(D) $\int_0^3 \left(5 - f(x)\right) dx + \int_3^6 \left(5 - g(x)\right) dx$

5. Consider the following sum:

$$\frac{1}{2\times4}+\frac{1}{3\times5}+\frac{1}{4\times6}+\ldots+\frac{1}{16\times18}$$

Which of the following expression is equivalent to the sum?

(A) $\sum_{n=1}^{8}\left(\frac{1}{(2n)(2n+2)}\right)$

(B) $\sum_{n=1}^{16}\frac{1}{(n)(n+2)}$

(C) $\sum_{n=3}^{16}\left(\frac{1}{(n-1)(n+1)}\right)$

(D) $\sum_{n=4}^{18}\left(\frac{1}{(n-2)n}\right)$

End of Section I

Section II

30 Marks
Attempt Questions 6 – 8
Allow about 48 minutes.

Answer each question in a SEPARATE writing booklet. Extra writing paper is available.
In Questions 6 – 8, your responses should include relevant mathematical reasoning and/or calculations.

Question 6 (11 Marks) Use a separate writing booklet.

(a) Find $\int 5x\sqrt{4-x^2}\,dx$ using the substitution $u = 4 - x^2$. **3**

(b) Use the substitution $x = u^2 - 2$ to evaluate $\int_2^7 \frac{x}{(2+x)^3}\,dx$ **3**

(c) Prove by mathematical induction that $8^n - 5^n$ is divisible by 3 for all positive integers n. **3**

(d) Find the equation of the function $f(x)$ if the gradient function is $f'(x) = 6x^2 - 4x + \frac{2}{x^2}$ and $f(2) = 1$. **2**

End of Question 6

Question 7 (10 marks) Use a SEPARATE writing booklet

(a) (i) Find an expression for the sum of the first n terms of the arithmetic series shown below. **2**

$$3+5+7+...+(2n+1)$$

(ii) Prove by mathematical induction that for all positive integers n **3**

$$1\times 2+2\times 3+3\times 4+...+n(n+1)=\frac{n(n+1)(n+2)}{3}.$$

(iii) Hence, find an expression for the sum of the first n terms of the series: **2**

$$7+17+31+49+71+...$$

(b) The first and fifth terms of an arithmetic series are the first two terms of a geometric series that has a common ratio of 2.

The twenty-first term of the arithmetic series is 72.

Find the first term and the common difference of the arithmetic series. **3**

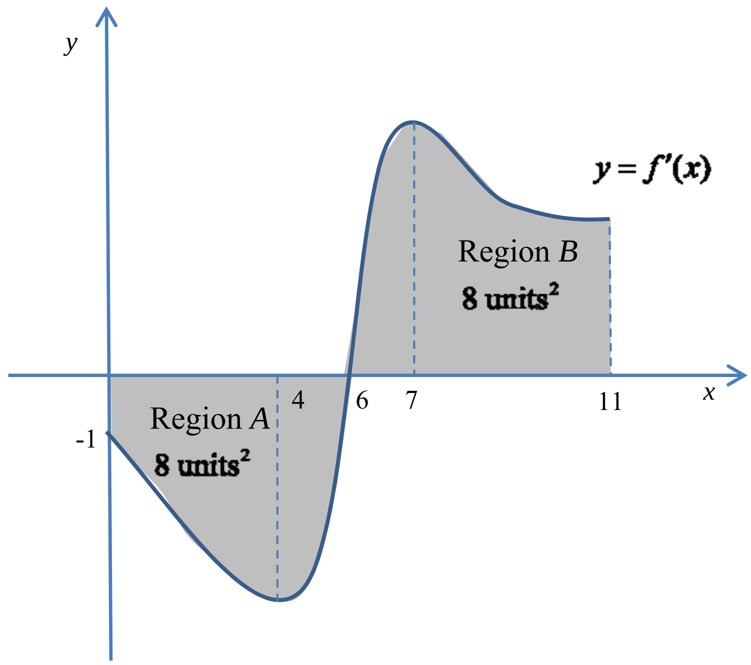
End of Question 7

Question 8 (10 marks) Use a SEPARATE writing booklet

(a) (i) Differentiate $\dfrac{3x^2+6}{2x+1}$. **1**

(ii) Hence, find $\displaystyle\int \frac{(x+2)(x-1)}{(2x+1)^2}\,dx$. **2**

(b) Consider the graph of $y = f'(x)$ for $0 \le x \le 11$ below. The region A and region B have the same area of 8 square units .



Given that $f(0) = 0$:

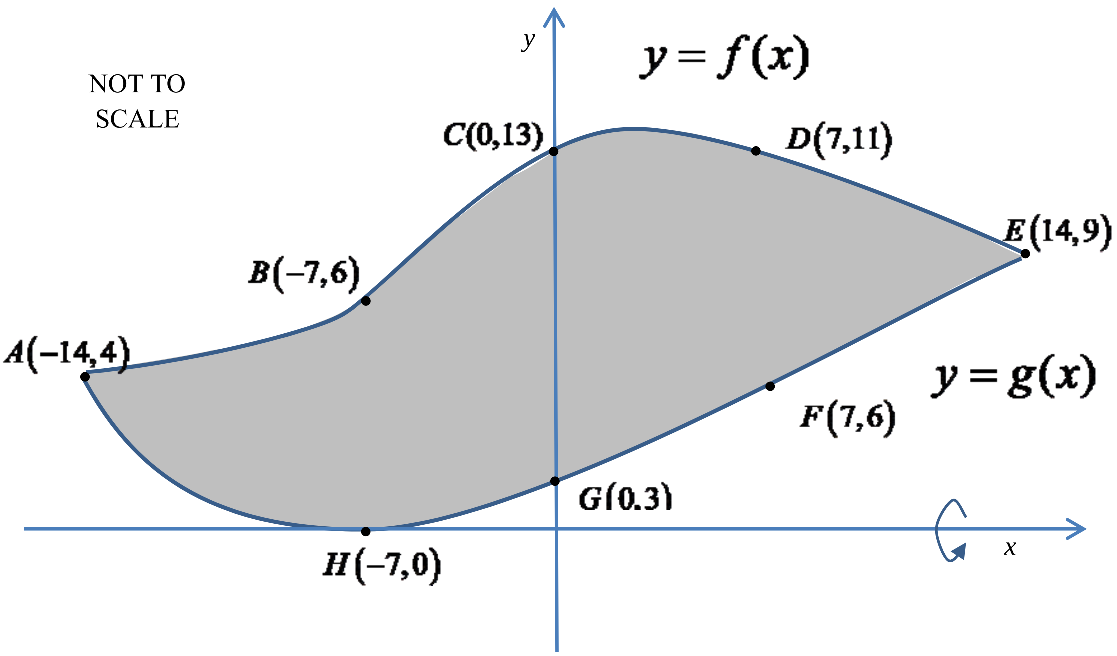
(i) Show that the graph of $y = f(x)$ has a turning point at $(6, -8)$ **2**

(ii) Graph $y = f(x)$ for $0 \le x \le 11$. **2**

Question 8 continues on the next page.

(c) Consider the graphs of $y = f(x)$ and $y = g(x)$ below. The two graphs intersect at points A and E. The points B, C and D lie on $y = f(x)$ and the points F, G and H lie on $y = g(x)$.

The area between the two graphs has been shaded.



Use the Trapezoidal Rule with five function values to find an estimate for the volume of the solid generated by rotating the shaded region around the x-axis. **3**

End of Examination

Multiple Choice

1. (D) $\frac{1}{3}\int_4^6 u^{-\frac{1}{2}}du$

2. (C) $x<-1, \quad x>1$

3. (C) -252

4. (D) $\int_0^3 (5-f(x))\,dx \;+\int_3^6 (5-g(x))\,dx$

5. (C) $\sum_{n=3}^{17}\left(\frac{1}{(n-1)(n+1)}\right)$

OR

(D) $\sum_{n=4}^{18}\left(\frac{1}{(n-2)n}\right)$

Question 6

(a) Find $\int 5x\sqrt{4-x^2}\,dx$ using the substitution $u=4-x^2$. **3**

$$\int 5x\sqrt{4-x^2}\,dx = -\frac{5}{2}\int \sqrt{u}\,du$$
$$= -\frac{5}{2}\int u^{\frac{1}{2}}\,du$$
$$= -\frac{5}{2}\left(\frac{2u^{\frac{3}{2}}}{3}\right)+C$$
$$= \frac{-5\sqrt{(4-x^2)^3}}{3}+C$$

$u=4-x^2$.

$$\frac{du}{dx}=-2x$$
$$dx=-\frac{du}{2x}$$
$$5x\sqrt{4-x^2}\,dx=-5x\sqrt{u}\frac{du}{2x}$$
$$=-\frac{5}{2}\sqrt{u}\;du$$

Generally well done. Several students mix x and u in the integrand. Whilst this was not penalised, this is not good setting out. Either have the integrand completely in *x* or completely in *u*. It may help to find an expression for the integrand in terms of *u* before replacing it in the integral. Don't forget the constant of integration for an indefinite integral.

(b) Use the substitution $x = u^2 - 2$ to evaluate $\int_2^7 \frac{x}{(2+x)^3}\,dx$ **3**

$$\int_2^7 \frac{x}{(2+x)^3}\,dx = \int_2^3 \frac{u^2-2}{(u^2)^3}(2u\,du)$$
$$= 2\int_2^3 \frac{u^3-2u}{u^6}\,du$$
$$= 2\int_2^3 \frac{u^3}{u^6} - \frac{2u}{u^6}\,du$$
$$= 2\int_2^3 \left(u^{-3} - 2u^{-5}\right)du$$
$$= 2\left[\frac{u^{-2}}{-2} - \frac{2u^{-4}}{-4}\right]_2^3$$
$$= \left[\left(\frac{1}{4} - \frac{1}{16}\right) - \left(\frac{1}{9} - \frac{1}{81}\right)\right]$$
$$= \frac{115}{1296}$$

$x = u^2 - 2$ Also $x + 2 = u^2$

$$\frac{dx}{du} = 2u$$
$$dx = 2u\,du$$

When $x = 7$ $u = 3$
When $x = 2$ $u = 2$

Again, mostly well done. Students are encouraged to do the working relating to changing of the integrand and limits to *u*, on the right hand side of the page. Separate this working clearly from the main working, by drawing a line if necessary. Nearly all students remembered to change the limits, however, some errors occurred in computing the new values and incurred in a minor penalty. Again do not mix x and u in the integrand. Common errors were having u^3 in the denominator of the integrand rather than $(u^2)^3 = u^6$. This simplified the following integration and evaluation and could not be awarded full carry on marks. Change negative indices to positive indices first to make manual evaluation easier. Show the substitution step when evaluating the primitive at upper and lower limits and then use a calculator!!

(c) Prove true for $n = 1$

$$8^1 - 5^1 = 8 - 5$$
$$= 3$$

Whish is divisible by 3
Therefore True for $n = 1$

Assume true for $n = k$

$8^k - 5^k = 3M$ Where M is an Integer

OR $8^k = 3M + 5^k$

Prove true for $n=k+1$ if true for $n=k$

Aim: Prove that:

$$8^{k+1}-5^{k+1}=3N \qquad \text{Where } N \text{ is an Integer}$$

$$\begin{aligned}
LHS &= 8^{k+1}-5^{k+1} \\
&= 8\times 8^k - 5\times 5^k \\
&= 8\left(3M+5^k\right)-5\times 5^k && \text{By Assumption} \\
&= 24M + 8\times 5^k - 5\times 5^k \\
&= 24M + 3\times 5^k \\
&= 3\left(8M+5^k\right) \\
&= 3N && \text{Since both } M \text{ and } k \text{ are both integers } \left(8M+5^k\right) \text{is an integer} \\
&= RHS
\end{aligned}$$

Therefore True for $n=k+1$ if true for $n=k$

Therefore, by mathematical induction 8^n-5^n is divisible by 3 for all positive integers n.

Mostly well done. You need to state <u>where</u> (ie in which step of the proof) you are using the assumption step. Note that the word "divisible" or "multiple" implies integer multiples and this needs to be stated eg ($8^k-5^k=3M$, where M is an integer). Several students did not bother making a proper conclusion after showing the result true for $n=k+1$. Whilst this was not penalised, a proof requires a convincing argument and you would hardly abandon the argument just when you are about to drive home the point!

(d)

$$f'(x)=6x^2-4x+2x^{-2}$$

$$\begin{aligned}
f(x) &= \frac{6x^3}{3}-\frac{4x^2}{2}+\frac{2x^{-1}}{-1}+C \\
&= 2x^3-2x^2-\frac{2}{x}+C
\end{aligned}$$

$$f(2)=2(2)^3-2(2)^2-\frac{2}{(2)}+C=1$$

$$16-8-1+C=1$$

$$7+C=1$$

$$C=-6$$

$$f(x)=2x^3-2x^2-\frac{2}{x}-6$$

Very well done. Most students earned the full marks here barring a few who made careless errors. $f(2)=1$ implies, when $x=2$, $f(x)=1$ not vice versa.

Question 7

(a) (i) $3+5+7+...+(2n+1)$ is an arithmetic series with first term 3 and common difference of 2

$$3+5+7+...+(2n+1)=\frac{n}{2}\left[3+(2n-1)\right]$$
$$=\frac{n}{2}\left[2n+2\right]$$
$$=n(n+1)$$

Generally well done. Still some mistakes with formula so use the reference sheet if you need it.

(ii) Prove true for $n=1$

$$LHS=1\times 2 \qquad RHS=\frac{1(1+1)(1+2)}{3}$$
$$=2 \qquad =\frac{6}{3}$$
$$=2$$

Therefore True for $n=1$

Assume true for $n=k$

$$1\times 2+2\times 3+3\times 4+...+k(k+1)=\frac{k(k+1)(k+2)}{3}$$

Prove true for $n=k+1$ if true for $n=k$

Aim: Prove that $1\times 2+2\times 3+3\times 4+...+k(k+1)+(k+1)(k+2)=\frac{(k+1)(k+2)(k+3)}{3}$

$LHS=1\times 2+2\times 3+3\times 4+...+k(k+1)+(k+1)(k+2)$

$=\frac{k(k+1)(k+2)}{3}+(k+1)(k+2)$ By Assumption

$=(k+1)(k+2)\left(\frac{k}{3}+1\right)$

$=\frac{(k+1)(k+2)}{3}(k+3)$

$=RHS$

Therefore True for $n=k+1$ if true for $n=k$

Therefore, by mathematical induction $1\times 2+2\times 3+3\times 4+...+n(n+1)=\frac{n(n+1)(n+2)}{3}$ is true for all positive integers n.

Mostly well done. This is a proof so reasoning needs to be fully communicated.
Explain reasoning from one line to the next.

(iii) $7+17+31+49+71+\ldots=\left[2\left(1\times2\right)+3\right]+\left[2\left(2\times3\right)+5\right]+\left[2\left(3\times4\right)+7\right]+\ldots$

The sum of the this series is twice sum in part b) plus sum in part a)

$$S_n=2\left(\frac{n(n+1)(n+2)}{3}\right)+n(n+1)$$
$$=\frac{2n^2(n+1)^2(n+2)}{3}$$

Some students found this challenging. The use of the word 'Hence' in this question gives a clue that this series may be based on a combination of parts i. and ii. If you developed a correct expression for the sum then subsequent algebraic errors were ignored.

(b)

$T_1=a$ and $T_5=a+4d$

$$\frac{T_5}{T_1}=\frac{a+4d}{a}$$
$$\frac{a+4d}{a}=2$$
$$a+4d=2a$$
$$4d=a$$

$$T_{21}=72$$
$$a+20d=72$$
$$a=72-20d$$

$$4d=72-20d$$
$$24d=72$$
$$d=3$$

$$a=4\left(3\right)$$
$$=12$$

First term is 12 and common difference is 3

Mostly well done. A good place to start was to write down using mathematical notation the information that is given in words in the question.

Read carefully. A number of students read 'twenty-first' as 'twenty-fifth'

A general comment for all of question 7 is that students need to keep their work legible and well set out. It is not the marker's job to guess what you know or assume you know thing that you have not communicated in your answer. You also don't get the luxury of appealing if something is missed by a marker because of poor setting out or illegible hand writing in the HSC.

Question 8

(a) (i)

$$\frac{d}{dx}\left(\frac{3x^2+6}{2x+1}\right) = \frac{(2x+1)\cdot\frac{d}{dx}(3x^2+6)-(3x^2+6)\cdot\frac{d}{dx}(2x+1)}{(2x+1)^2}$$

$$= \frac{(2x+1)\cdot 6x - (3x^2+6)\cdot 2}{(2x+1)^2}$$

$$= \frac{6x^2+6x-12}{(2x+1)^2}$$

$$= \frac{6(x+2)(x-1)}{(2x+1)^2}$$

** No general issues

(ii)

$$\int \frac{(x+2)(x-1)}{(2x+1)^2}dx = \frac{1}{6}\int \frac{6(x+2)(x-1)}{(2x+1)^2}dx$$

$$= \frac{1}{6}\cdot\frac{3x^2+6}{2x+1}+c \qquad \text{[from part (i)]}$$

$$= \frac{x^2+2}{2(2x+1)}+c$$

** Issue: Remember the $+c$

(b) (i) From graph of $y = f'(x)$, $f'(6) = 0$.

$\therefore$ $y = f(x)$ has a stationary point at $x = 6$

Since $f(6^-) < 0$ and $f(6^+) > 0$, this is a minimum turning point

$$\int_0^6 f'(x)dx = -8$$

$$\left[f(x)\right]_0^6 = -8$$

$$f(6) - f(0) = -8$$

$$f(6) = -8 \qquad [\text{since } f(0) = 0]$$

** Issues:

(1) Many said that the above integral was equal to 8 instead of -8.

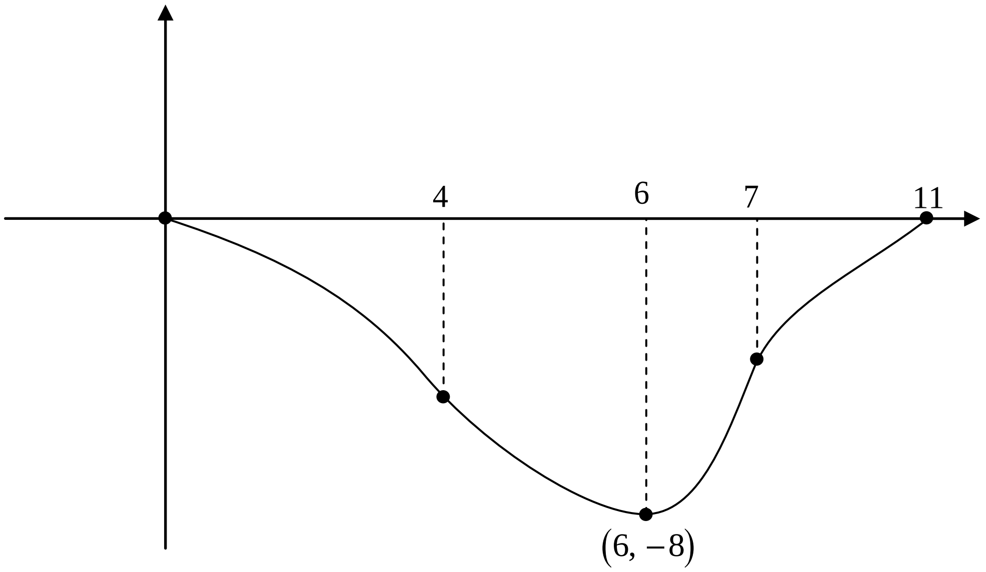
(2) The (signed) area under the x-axis does not generally give the y-value – it gives the **change** in the y-value. So for those who tried to argue with words, you needed to mention that $f(0) = 0$ so that the change in y is equal to the y value **in this particular example**.

(ii) Just as the y-value decreases by 8 from $x=0$ to $x=6$, it increases by 8 from $x=6$ to $x=11$.

Using mathematical notation:

$$\begin{aligned}\delta y &= \int_0^6 f'(x)\,dx + \int_6^{11} f'(x)\,dx \\ &= -8+8 \\ &= 0\end{aligned}$$

So since $f(0)=0$, then $f(11)=0$



** Issues: (1) not starting the graph at the origin since $f(0)=0$

(2) not finishing the graph at $(11,0)$ since $f(11)=0$

(3) not having points of inflexion, and not indicating their x-values when present

(4) When inflexions were present, having the wrong concavity either side of those inflexions. The gradient function is getting more negative at x changes from 0 to 6, so the curve should be getting **steeper**.

(c)

$$\begin{aligned}V &= \pi\int_{-14}^{14}\left[f(x)\right]^2 dx - \pi\int_{-14}^{14}\left[g(x)\right]^2 dx \\ &= \pi\int_{-14}^{14}\left\{\left[f(x)\right]^2-\left[g(x)\right]^2\right\}dx\end{aligned}$$

x	–14	–7	0	7	14
$\left[f(x)\right]^2-\left[g(x)\right]^2$	0	36	160	85	0

$$\begin{aligned}V &\approx \pi\cdot\frac{7}{2}\left[0+0+2(36+160+85)\right] \\ &= 1967\pi \text{ units}^3\end{aligned}$$

**Issues: (1) Squaring the difference instead of subtracting the squares.

(2) Squaring an area and thinking it gives a volume